

NEW INTEGRABLE EXTENSIONS OF $N=2$ KdV AND BOUSSINESQ HIERARCHIES

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Abstract

We construct a new variety of $N = 2$ supersymmetric integrable systems by junction of pseudo-differential superspace Lax operators for $a = 4$, $N = 2$ KdV and multi-component $N = 2$ NLS hierarchies. As an important particular case, we obtain Lax operator for $N = 4$ super KdV system. A similar extension of one of $N = 2$ super Boussinesq hierarchies is given. We also present a minimal $N = 4$ supersymmetric extension of the second flow of $N = 4$ KdV hierarchy and comment on its possible integrability.

1. Introduction. For the last years, $N = 2$ supersymmetric hierarchies of integrable equations (of the KP, KdV and NLS types) attracted an increasing interest, mainly due to their potential physical applications in non-perturbative $2D$ supergravity, superextensions of matrix models and topological field theories (see, e.g. [1]). Unexpected interrelations between these hierarchies were revealed and different manifestly $N = 2$ supersymmetric Lax representations for them were constructed [2-5].

In ref. [4] a new form of the Lax representation for the $a = 4$, $N = 2$ KdV hierarchy through the pseudo-differential Lax operator was proposed¹

$$L = \partial - 2J - 2\bar{D}\partial^{-1}(DJ) \quad , \quad (1)$$

$$\frac{\partial}{\partial t_k} L = [(L^k)_{\geq 1}, L] \quad . \quad (2)$$

Here, $J = J(z, \theta, \bar{\theta})$ is a general $N = 2$ superfield, $k = 1, 2, \dots$ and the subscript ≥ 1 means restriction to the purely differential part of L^k .

As distinct from the previously known representation with the differential Lax operator [6], this form produces the whole set of bosonic conserved quantities for the $a = 4$, $N = 4$ KdV (both of odd and even scale dimensions) by the general formula

$$H^n = \int dZ (L^n)_0 \quad . \quad (3)$$

Note a non-standard definition of the residue of the powers of the Lax operator in (3). In the conventional Lax representation [6], the standard definition of the residue as a coefficient before $[D, \bar{D}]\partial^{-1}$ is used, however, only odd-dimension conserved charges can be directly constructed within its framework.

The Lax representation (1), (2) is also advantageous in that it provides a link with the $N = 2$ NLS hierarchy. After the Miura type transformation [3]

$$J = -\frac{1}{2}F\bar{F} - \frac{1}{2}\frac{D\bar{F}'}{D\bar{F}}, \quad (4)$$

where F, \bar{F} are chiral and anti-chiral $N = 2$ superfields ($DF = \bar{D}\bar{F} = 0$), some further similarity transformation of the Lax operator and passing to its adjoint, one gets new Lax operator [4]

$$L' = \partial - F\bar{F} - F\bar{D}\partial^{-1}(D\bar{F}). \quad (5)$$

It gives rise, via the representation (2), to the minimal $N = 2$ extension of NLS hierarchy. In ref. [5] multi-component generalizations of the Lax operator (5) were constructed.

The above correspondence between two Lax operators generalizes the situation known in the purely bosonic case for the $R - S$ system. The bosonic analogs of the $N = 2$ Lax operators (1), (5) are as follows [7, 8]

$$L^{(1)} = \partial + R\frac{1}{\partial - S}, \quad (6)$$

¹We use the following conventions about the algebra of $N = 2$, $1D$ superspace derivatives

$$D = \frac{\partial}{\partial\theta} - \frac{1}{2}\bar{\theta}\frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z}, \quad \{D, \bar{D}\} = -\frac{\partial}{\partial z}, \quad \{D, D\} = \{\bar{D}, \bar{D}\} = 0,$$

$$(D)^\dagger = -\bar{D}, \quad (\frac{\partial}{\partial z})^\dagger = -\frac{\partial}{\partial z}.$$

$$L^{(2)} = \partial + r\partial^{-1}\bar{r} . \quad (7)$$

They are related to each other by a generalized Miura transformation

$$R = r\bar{r}, \quad S = \frac{\bar{r}'}{\bar{r}} . \quad (8)$$

One may construct a "hybrid" Lax operator as the sum $L^{(1)} + L^{(2)}$. However, the latter does not produce a new system as it can be reduced to a two-component generalization of (7) by the transformation (8) (a self-consistent generalization of (7) is to attach an index i to the field r and to sum up over i).

Since the $N = 2$ Lax operators (1), (5) are related in a rather obscure way (through an additional similarity transformation and conjugation), an analogous hybridization procedure in this case might yield new integrable systems, more general than the $N = 2$ KdV or $N = 2$ NLS hierarchies. The basic aim of the present note is to demonstrate that this is indeed the case. We also elaborate on some generalizations and consequences of this fact.

2. Hybrid $N = 2$ KdV-NLS hierarchies. Let us introduce M pairs of chiral and anti-chiral dimension $1/2$ superfields F^i, \bar{F}^i . They can be either fermionic or bosonic and in general are not obliged to be conjugated to each other (in this case J should also be complex). The $U(1)$ charges of F and \bar{F} inside each pair are taken to be opposite, but the relative charges of different pairs are noway fixed. We construct the following Lax operator

$$L_1 = \partial - 2J - 2\bar{D}\partial^{-1}(DJ) - \sum_i F^i \bar{D}\partial^{-1}(D\bar{F}^i) + \sum_i \bar{D}\partial^{-1}(D(F^i \bar{F}^i)) . \quad (9)$$

We have checked that it gives rise, through the same Lax equation (2), to the self-consistent hierarchy of the evolution equations. When $F^i = \bar{F}^i = 0$, the operator L_1 is reduced to (1). With $J = \frac{1}{2} \sum_i F^i \bar{F}^i$, the multi-component generalization of the Lax operator (5) is recovered [5]. Respectively, the related hierarchy is reduced to either $a = 4$, $N = 2$ KdV or multi-component NLS ones. Thus we have got new integrable extensions of both these hierarchies.

Explicitly, the second and third flows are as follows

$$\begin{aligned} \frac{\partial J}{\partial t_2} &= -[D, \bar{D}] J' - 4JJ' + \sum_i (\bar{D}F^i D\bar{F}^i)' , \\ \frac{\partial F^i}{\partial t_2} &= F^{i''} + 4D(J\bar{D}F^i) , \\ \frac{\partial \bar{F}^i}{\partial t_2} &= -\bar{F}^{i''} + 4\bar{D}(JD\bar{F}^i) , \\ \frac{\partial J}{\partial t_3} &= J''' + 3([D, \bar{D}] JJ)' + \frac{3}{2}([D, \bar{D}] J^2)' + 4(J^3)' - \frac{3}{2} \sum_i (F^{i'} \bar{F}^{i'} + 4J\bar{D}F^i D\bar{F}^i)' , \\ \frac{\partial F^i}{\partial t_3} &= F^{i'''} + D \left(6\bar{D}(JF^{i'}) - 12J^2 \bar{D}F^i + 3 \sum_j D\bar{F}^j \bar{D}F^j \bar{D}F^i \right) , \\ \frac{\partial \bar{F}^i}{\partial t_3} &= \bar{F}^{i'''} - \bar{D} \left(6D(J\bar{F}^{i'}) + 12J^2 D\bar{F}^i - 3 \sum_j D\bar{F}^j \bar{D}F^j D\bar{F}^i \right) . \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{\partial J}{\partial t_3} &= J''' + 3([D, \bar{D}] JJ)' + \frac{3}{2}([D, \bar{D}] J^2)' + 4(J^3)' - \frac{3}{2} \sum_i (F^{i'} \bar{F}^{i'} + 4J\bar{D}F^i D\bar{F}^i)' , \\ \frac{\partial F^i}{\partial t_3} &= F^{i'''} + D \left(6\bar{D}(JF^{i'}) - 12J^2 \bar{D}F^i + 3 \sum_j D\bar{F}^j \bar{D}F^j \bar{D}F^i \right) , \\ \frac{\partial \bar{F}^i}{\partial t_3} &= \bar{F}^{i'''} - \bar{D} \left(6D(J\bar{F}^{i'}) + 12J^2 D\bar{F}^i - 3 \sum_j D\bar{F}^j \bar{D}F^j D\bar{F}^i \right) . \end{aligned} \quad (11)$$

An interesting peculiarity of eqs. (10), (11) is that the dimension $1/2$ superfields F^i and \bar{F}^i appear in the nonlinear terms only under spinor derivatives, i.e. as $\Phi^i = \bar{D}F^i$, $\bar{\Phi}^i = D\bar{F}^i$,

$\overline{D}\Phi^i = D\bar{\Phi}^i = 0$ (for instance, $F^{i'} = -D\bar{\Phi}^i$, etc). Acting on both sides of the F equations by D, \overline{D} , one can rewrite the above sets entirely in terms of the chiral and anti-chiral dimension 1 superfields $\Phi^i, \bar{\Phi}^i$. In this sense F^i and \bar{F}^i can be regarded as prepotentials of the superfields $\Phi^i, \bar{\Phi}^i$ in some fixed gauge with respect to the prepotential gauge freedom. Note that the relation between the superfields Φ^i and F^i is invertible

$$F^i = -D \partial^{-1} \Phi^i, \quad \bar{F}^i = -\overline{D} \partial^{-1} \bar{\Phi}^i. \quad (12)$$

It is somewhat surprising that for one pair of mutually conjugated fermionic superfields $F, \bar{F} = F^\dagger$ and real J , the systems (10), (11), being rewritten in terms of the superfields $\Phi, \bar{\Phi}$, coincide with the second and third flows of the $N = 4, SU(2)$ KdV hierarchy constructed in [9, 10] (actually, for one of possible equivalent choices of the $SU(2)$ breaking parameters in it, $a = 4, b = 0$). For instance, the second flow equations take the form

$$\begin{aligned} \frac{\partial J}{\partial t_2} &= -[D, \overline{D}] J' - 4JJ' + (\Phi\bar{\Phi})' \quad , \\ \frac{\partial \Phi}{\partial t_2} &= \Phi'' + 4\overline{D}D(J\Phi) \quad , \\ \frac{\partial \bar{\Phi}}{\partial t_2} &= -\bar{\Phi}'' + 4D\overline{D}(J\bar{\Phi}) \quad . \end{aligned} \quad (13)$$

It is easy to check the covariance of this set under the transformations of an extra hidden $N = 2$ supersymmetry [10]

$$\delta J = \frac{1}{2}\epsilon D\Phi + \frac{1}{2}\bar{\epsilon}\overline{D}\bar{\Phi}, \quad \delta\Phi = -2\bar{\epsilon}\overline{D}J, \quad \delta\bar{\Phi} = -2\epsilon DJ. \quad (14)$$

Here, $\epsilon, \bar{\epsilon}$ are mutually conjugated Grassmann parameters. Together with the explicit $N = 2$ supersymmetry these transformations constitute $N = 4$ supersymmetry in one dimension. Note that an equivalent realization in terms of the superfields F, \bar{F} is non-local

$$\delta J = -\frac{1}{2}\epsilon F' - \frac{1}{2}\bar{\epsilon}\bar{F}', \quad \delta F = -2\bar{\epsilon}D\partial^{-1}\overline{D}J, \quad \delta\bar{F} = -2\epsilon\overline{D}\partial^{-1}DJ. \quad (15)$$

Thus in this particular case the Lax equation (2) solves the problem of constructing the Lax representation for $N = 4$ KdV hierarchy. Actually, this proves the very existence of such an integrable hierarchy, the fact conjectured in [9, 10] on the ground of the existence of higher order non-trivial conservation laws and the bi-hamiltonian property for this system. Now it is a matter of straightforward computation, using the general formula (3), to reproduce the conserved quantities which were constructed in [10] by the "brute force" method.

Note that another choice of the relation between F and \bar{F} ,

$$\bar{F} = -F^\dagger, \quad (\bar{\Phi} = -\Phi^\dagger) \quad (16)$$

(with keeping J real as before) results in a different system. It formally coincides with (13) but is invariant under the following modification of the transformations (17)

$$\delta J = \frac{1}{2}\epsilon D\Phi - \frac{1}{2}\bar{\epsilon}\overline{D}\bar{\Phi}, \quad \delta\Phi = 2\bar{\epsilon}\overline{D}J, \quad \delta\bar{\Phi} = -2\epsilon DJ. \quad (17)$$

Together with the manifest $N = 2$ supersymmetry these constitute a "twisted" $N = 4$ supersymmetry (commutator of two such transformations yields $\frac{\partial}{\partial z}$ with the opposite sign as

compared to the manifest $N = 2$ supersymmetry transformations and the transformations (14) or (15)).

For a greater number of pairs F, \bar{F} we get the extensions which, to our knowledge, were not considered before. The bosonic sector of the generic system (10) reads

$$\begin{aligned}\frac{\partial J}{\partial t_2} &= -T' - 4JJ' - \sum_i (\bar{H}^i H^i)' , \\ \frac{\partial T}{\partial t_2} &= -J''' + \sum_i (\bar{H}^i H^{i'} - \bar{H}^{i'} H^i)' - 4(TJ)' , \\ \frac{\partial H^i}{\partial t_2} &= -H^{i''} + 2TH^i - 4JH^{i'} - 2J'H^i , \\ \frac{\partial \bar{H}^i}{\partial t_2} &= \bar{H}^{i''} - 2T\bar{H}^i - 4J\bar{H}^{i'} - 2J'\bar{H}^i .\end{aligned}\tag{18}$$

Here the bosonic components are defined as

$$J = J|, T = [D, \bar{D}]J|, H = -D\bar{F}|, \bar{H} = \bar{D}F| \tag{19}$$

and $|$ means the restriction to the $\theta = \bar{\theta} = 0$ parts.

3. An extension of $N = 2$ Boussinesq hierarchy. One may wonder whether similar extensions are possible for generalized $N = 2$ KdV hierarchies associated with $N = 2$ W_n algebras as the second hamiltonian structures. In refs. [11, 12] $N = 2$ superfield differential Lax operator for the $\alpha = -1/2$, $N = 2$ Boussinesq equation (with $N = 2$ W_3 the second hamiltonian structure) was constructed

$$L = D \left(\partial^2 - 3J\partial - T - \frac{3}{2}J' - \frac{1}{2}[D, \bar{D}]J + 2J^2 \right) \bar{D} . \tag{20}$$

In trying to modify it along the above lines we have found that the only consistent modification is the following one

$$L_1 = D \left(\partial^2 - 3J\partial - T - \frac{3}{2}J' - \frac{1}{2}[D, \bar{D}]J + 2J^2 + \sum_i \Psi^i \partial^{-1} \bar{\Psi}^i \right) \bar{D} \tag{21}$$

where M pairs of fermionic (bosonic) chiral and anti-chiral superfields $\Psi^i, \bar{\Psi}^i$ with dimension $3/2$ were introduced. This new Lax operator (21) gives rise through the Lax equation

$$\frac{\partial L_1}{\partial t_2} = \left[\left(L_1^{\frac{2}{3}} \right)_{>1}, L_1 \right] \tag{22}$$

to the following extension of $N = 2$ Boussinesq hierarchy

$$\begin{aligned}\frac{\partial J}{\partial t_2} &= 2T' + [D, \bar{D}]J' - 2JJ' , \\ \frac{\partial T}{\partial t_2} &= -2J''' + 10(\bar{D}JDJ)' + 4J'[D, \bar{D}]J + 2J[D, \bar{D}]J' + 4J'J^2 + 6\bar{D}JDT + \\ &\quad 6DJ\bar{D}T + 6J'T + 2JT' - 6\sum_i (\Psi^i \bar{\Psi}^i)' , \\ \frac{\partial \Psi^i}{\partial t_2} &= 3\Psi^{i''} - 6J\Psi^{i'} + 6DJ\bar{D}\Psi^i , \\ \frac{\partial \bar{\Psi}^i}{\partial t_2} &= -3\bar{\Psi}^{i''} - 6J\bar{\Psi}^{i'} + 6\bar{D}JD\bar{\Psi}^i .\end{aligned}\tag{23}$$

We checked the existence of first non-trivial higher-order conservation laws for this hierarchy by using the general formula

$$H_k = \int dX \text{Res} L^{k/3} \quad (24)$$

(in this case, just as in the $\alpha = -1/2$, $N = 2$ Boussinesq limit $\Psi^i = \bar{\Psi}^i = 0$, the residue of $N = 2$ pseudo-differential operators is defined in the standard way as the coefficient before $[D, \bar{D}] \partial^{-1}$). We found that there exist conserved quantities H_k of *all* scale dimensions k , while in the pure $\alpha = -1/2$, $N = 2$ Boussinesq case H_{3n} drop out [11]. The evident reason for non-existence of H_{3n} in this case and their presence in the modified case is that the $N = 2$ Boussinesq Lax operator (20) is differential and so is its any integer power, while (21) is pseudo-differential. One can check by explicit computation that the densities \mathcal{H}_k , $k = 3n$ vanish when $\Psi^i, \bar{\Psi}^i$ are equated to zero.

It is an open question whether one can consistently reduce the above hierarchy to the form containing only extra superfields $\Psi^i, \bar{\Psi}^i$, similarly to the case of extended $N = 2$ KdV hierarchy discussed in the previous section (recall that it is achieved by putting $J = \frac{1}{2} \sum_i F^i \bar{F}^i$ in eqs. (10), (11)). An essential difference of the set (23) from (10), (11) consists in that in the former case it is impossible to trade $\Psi^i, \bar{\Psi}^i$ for their spinor derivatives by applying the latter to both sides of the Ψ equations (actually, these superfields are present on their own in the equation for T as well).

4. A new $N = 4$ supersymmetric system. As the last remark, let us present a minimal $N = 4$ supersymmetry preserving extension of the second $N = 4$ KdV flow (13).

The simplest possibility to extend the set J, F, \bar{F} to some reducible $N = 4$ supermultiplet is to add two extra pairs of mutually conjugated superfields F_i, \bar{F}_i , $i = 1, 2$ (for definiteness, we choose them fermionic) with the following transformation law under the hidden $N = 2$ supersymmetry

$$\delta F_1 = -\bar{\epsilon} D \bar{F}_2, \quad \delta F_2 = \bar{\epsilon} D \bar{F}_1, \quad \delta \bar{F}_1 = -\epsilon \bar{D} F_2, \quad \delta \bar{F}_2 = \epsilon \bar{D} F_1, \quad (25)$$

or, in terms of $\Phi_i = \bar{D} F_i, \bar{\Phi}_i = D \bar{F}_i$,

$$\delta \Phi_1 = \bar{\epsilon} \bar{D} \bar{\Phi}_2, \quad \delta \Phi_2 = -\bar{\epsilon} \bar{D} \bar{\Phi}_1, \quad \delta \bar{\Phi}_1 = \epsilon D \Phi_2, \quad \delta \bar{\Phi}_2 = -\epsilon D \Phi_1. \quad (26)$$

It is easy to check that the Lie bracket of these transformations is the same as for (14), (15).

The second flow system (10) with three extra pairs of the F superfields, as it stands, does not respect covariance under (15), (25). However, let us consider the following modification of it (in the notation through $\Phi, \bar{\Phi}, \Phi_i, \bar{\Phi}_i$)

$$\begin{aligned} \frac{\partial J}{\partial t_2} &= -[D, \bar{D}] J' - 4JJ' + (\Phi \bar{\Phi})' - (\Phi_1 \bar{\Phi}_1)' + (\Phi_2 \bar{\Phi}_2)' , \\ \frac{\partial \Phi}{\partial t_2} &= \Phi'' + 4\bar{D} D (J\Phi + \frac{1}{2}\Phi_1 \Phi_2) , \\ \frac{\partial \bar{\Phi}}{\partial t_2} &= -\bar{\Phi}'' + 4D \bar{D} (J\bar{\Phi} + \frac{1}{2}\bar{\Phi}_1 \bar{\Phi}_2) , \\ \frac{\partial \Phi_1}{\partial t_2} &= \Phi_1'' + 4\beta \bar{D} D (J\Phi_1 - \frac{1}{2}\Phi \bar{\Phi}_2) , \\ \frac{\partial \bar{\Phi}_1}{\partial t_2} &= -\bar{\Phi}_1'' + 4\beta D \bar{D} (J\bar{\Phi}_1 - \frac{1}{2}\bar{\Phi} \Phi_2) , \end{aligned}$$

$$\begin{aligned}
\frac{\partial \Phi_2}{\partial t_2} &= -\Phi_2'' + 4\beta \overline{D}D (J\Phi_2 + \frac{1}{2}\Phi\bar{\Phi}_1) \quad , \\
\frac{\partial \bar{\Phi}_2}{\partial t_2} &= \bar{\Phi}_2'' + 4\beta D\overline{D} (J\bar{\Phi}_2 + \frac{1}{2}\bar{\Phi}\Phi_1) \quad ,
\end{aligned} \tag{27}$$

β being a parameter. One immediately checks that the modified system is $N = 4$ supercovariant. Note that the extra superfields Φ_i have the dimension 1, just as Φ , but their $U(1)$ charges are twice as smaller compared to the $U(1)$ charge of Φ .

For the time being, we did not succeed in finding a modification of the Lax operator (9) which would lead to (27) via the equation like (2). Instead we studied the issue of existence of higher-order non-trivial conserved quantities for (27), as the standard test for integrability. Using the undetermined coefficients method and the Mathematica package for $N = 2$ superfield computations [13], we found that at least two non-trivial conserved charges exist for this system

$$\begin{aligned}
H_2 &= \int dX \left\{ J^2 - \frac{1}{2}\Phi\bar{\Phi} + \frac{1}{2\beta} (\Phi_1\bar{\Phi}_1 - \Phi_2\bar{\Phi}_2) \right\} \quad , \\
H_3 &= \int dX \left\{ \frac{2}{3}J^3 + \overline{D}JDJ + \frac{1}{4}\Phi'\bar{\Phi} - \frac{1}{4\beta} (\Phi_1'\bar{\Phi}_1 + \Phi_2'\bar{\Phi}_2) - J\Phi\bar{\Phi} \right. \\
&\quad \left. + J\Phi_1\bar{\Phi}_1 - J\Phi_2\bar{\Phi}_2 - \frac{1}{2}\Phi\bar{\Phi}_1\bar{\Phi}_2 - \frac{1}{2}\bar{\Phi}\Phi_1\Phi_2 \right\} \quad .
\end{aligned} \tag{28}$$

These quantities respect rigid $N = 4$ supersymmetry and, after setting $\Phi_i = \bar{\Phi}_i = 0$ and further $\Phi = \bar{\Phi} = 0$, are reduced to the same dimension conserved charges of the $N = 4$, $SU(2)$ KdV and $a = 4$, $N = 2$ KdV, respectively.

We also analyzed the existence of the next conserved hamiltonian, H_4 . We found that no such quantity exists, provided the relevant density is local in the superfields $\Phi_i, \bar{\Phi}_i$. Recall, however, that the basic objects we started with are the dimension 1/2 fermionic superfields F_i, \bar{F}_i . We conjecture that the candidate higher-order conserved quantities, beginning with H_4 , should include terms where these basic superfields appear on their own, with no spinor derivatives on them. These terms are *nonlocal* when written in terms of $\Phi_i, \bar{\Phi}_i$. Another possibility is that similar terms could be inserted as well in eqs. (27) with preserving $N = 4$ supersymmetry. We will elaborate on these possibilities elsewhere.

It would be interesting to find the Lax operator (if existing) and the hamiltonian formulation for the above system, including the second hamiltonian structure superalgebra. We suspect that this system (or some its modification) could bear a tight relation to the super KdV hierarchy with the “large” $N = 4$, $SO(4) \times U(1)$ superconformal algebra as the second hamiltonian structure. Indeed, inspecting the component contents of the relevant superfield set $J, \Phi, \bar{\Phi}, F_i, \bar{F}_i$, $i = 1, 2$, we find four dimension 1/2 and four dimension 3/2 fermionic fields, as well as seven dimension 1 and one dimension 2 bosonic fields. This is just the currents contents of the “large” $N = 4$ superconformal algebra [14, 15].

5. Conclusions. In this Letter we constructed new $N = 2$ supersymmetric integrable systems by junction of the pseudo-differential superspace Lax operators for $a = 4$, $N = 2$ KdV and multi-component $N = 2$ NLS hierarchies. As a by-product we obtained Lax operator for $N = 4$, $SU(2)$ super KdV system and thus proved the integrability of the latter. A similar extension of the $\alpha = -1/2$, $N = 2$ super Boussinesq hierarchy was found.

An intriguing characteristic feature of the proposed construction is the possibility to extend some particular Lax operator by M additional $N = 2$ chiral and anti-chiral superfields. We

are still not aware of the general recipe of how to construct such extensions, only two above examples have been explicitly worked out so far. Now it is under investigation whether the remaining two $N = 2$ KdV and Boussinesq hierarchies (the $a = -2, 1$ KdV and $\alpha = -2, 5/2$ Boussinesq ones) can be extended in a similar way.

It seems also very interesting to study in more detail the $N = 4$ supersymmetric extension of the second flow of $N = 4$ KdV hierarchy and to check its possible integrability. There remains a problem of putting this system, as well as the above Lax representation for $N = 4$ KdV, into a manifestly $N = 4$ supersymmetric form (e.g., in the framework of $1D$ harmonic superspace).

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